

Ambiguity Effects in Asset Market Equilibrium under Two-Tiered Asymmetric Information*

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The paper investigates how the ‘degree of ambiguity’ affects liquidity risk, expected price sensitivity, market depth, and price volatility in asset markets. We analyze asset market equilibrium under two-tiered asymmetric information by introducing uninformed investors with ambiguity into the model of Grossman and Stiglitz (1980) without endogenous information acquisition.

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1. INTRODUCTION

Information plays a key role in determining investors' choices in financial markets. Investors always try to collect more information from various sources for making better investment decisions. According to their status or experiences, some investors could easily access to core information about future payoffs, while others could have only ambiguous information about them. Moreover, uninformed investors could refine their information by observing asset prices, which contain the information of informed investors. Presumably, information is asymmetrically distributed over investors and the qualities of information which investors acquire are different.

Asset pricing under asymmetric information have been extensively studied for several decades. In their seminal paper, Grossman and Stiglitz (1980) examine rational expectations equilibrium under the assumption that uninformed investors know only the distribution of a risky asset's true value.¹⁾ Recently, however, financial crises have motivated economists to pay more attention to ambiguity (ambiguous information), which represents uncertainty about the distribution. During financial crises, investors are likely to have ambiguous information and thus make cautious investments in financial markets. This leads to market illiquidity that exacerbates market situations. Therefore, ambiguous information is another important factor which affects asset market equilibrium.

To understand the roles of asymmetric and ambiguous information in financial markets, we incorporate them into the model of Grossman and Stiglitz (1980) by dividing uninformed investors into investors with and without ambiguity. Whole investors consist of informed and uninformed investors. Informed investors have information about the true value of a risky asset while the uninformed know only its distribution and observe asset price. Uninformed investors are divided into non-ambiguous and ambiguous investors. The former have a single belief (distribution) about

¹⁾ Grossman (1976) studies asset market equilibrium when all investors are uninformed with heterogeneous beliefs. His model is extended by Hellwig (1980), Diamond and Verrecchia (1981), Verrecchia (1981), and Admati (1985) among others.

future payoffs and are standard expected utility maximizers, while the latter have multiple beliefs about them and thus are maximin expected utility maximizers in the sense of Gilboa and Schmeidler (1989). This scheme leads to two-tiered asymmetric information: the first tier lies between informed and uninformed investors; the second tier between non-ambiguous and ambiguous investors. Thus our model involves more diverse qualities of information than Grossman and Stiglitz (1980).

The purpose of the paper is to investigate the effects of ambiguous information on asset prices in rational expectations equilibrium under two-tiered asymmetric information, when there is no endogenous information acquisition. The two-tiered asymmetric information structure enables us to conduct a rich analysis of ambiguity effect on asset market equilibrium. More precisely, we make comparative statics on equilibrium by considering the degree of ambiguity both at the individual and the market levels. Here the market-level degree of ambiguity is represented by the the population of ambiguous investors among uninformed investors.²⁾ In particular, we analyze how the ‘degree of ambiguity’ affects liquidity risk, expected price sensitivity, market depth, and price volatility in asset market equilibrium.

We find that there exists a unique rational expectations equilibrium where ambiguous investors do not participate in some intermediate price region. In other words, ambiguity-averse investors take positions only when the equilibrium price are sufficiently low or high. If ambiguity is present, the equilibrium price responds more sensitively to supply shock, which reduces market depth. Furthermore, if the individual or market degree of ambiguity increases, then price volatility would increase. Liquidity risk is also increasing in the individual degree of ambiguity. But for the case of the market degree of ambiguity, it shows such property only when the individual degree of ambiguity is sufficiently large (See Proposition 1). This result is remarkable because we expect that ambiguity will increase market illiquidity.

Recently, finance literature began to focus on asset prices under

²⁾ This is for simplifying the arguments since the population of informed investors are fixed. In fact it is the population of ambiguous investors.

asymmetric and ambiguous information. Along the line of Grossman and Stiglitz (1980), Mele and Sangiorgi (2011) and Ozsoylev and Werner (2011) study asset markets where there exist identical informed investors and uninformed investors with ambiguity. Based on Grossman (1976), Cao *et al.* (2005), Easley and O'Hara (2009, 2010), and Ui (2011) investigate asset market equilibrium when investors have heterogeneous ambiguous information about the distribution of asset payoffs.

This paper is closely related to Mele and Sangiorgi (2011) and Ozsoylev and Werner (2011). However, our model is differentiated from theirs in two respects. First, while every uninformed investor faces ambiguity in Mele and Sangiorgi (2011) and Ozsoylev and Werner (2011), ambiguous and non-ambiguous traders may coexist in our model. In other words, we refine the information quality of uninformed investors by dividing them into investors with and without ambiguity. Second, we consider both individual and market degree of ambiguity as variable ambiguity parameters. Ozsoylev and Werner (2011) assume that there are a representative informed investor and a representative uninformed investor with ambiguity.³⁾ Thus their model is not subject to population changes of informed or ambiguous investors.

In this paper, we assume that all the investors are risk averse, in particular, have CARA utility, as in Cao *et al.* (2005).⁴⁾ Moreover, ambiguous investors have multiple beliefs only about the mean of the risky asset's true value with exact information about its variance, as in Cao *et al.* (2005) and Ui (2010).⁵⁾ This assumption allows us to simplify the characterization of asset market equilibrium without loss of generality.⁶⁾

³⁾ Epstein and Schneider (2008) and Illeditsch (2011) assume that there is a single representative investor.

⁴⁾ See also Illeditsch (2011), Mele and Sangiorgi (2011) and Ui (2010). In contrast, Epstein and Schnieder (2008) and Ozsoylev and Werner (2011) consider ambiguous investors is risk neutral.

⁵⁾ Epstein and Schnieder (2008) and Illeditsch (2011) assume that there exists ambiguous information only about variances of observed signals. In general, ambiguous investors may know neither the mean nor the variance as in Easley and O'Hara (2009, 2010) and Ozsoylev and Werner (2011).

⁶⁾ It is worth noting that illiquidity phenomenon disappears in Ozsoylev and Werner (2011)

The rest of the paper is organized as follows. In section 2, we introduce the model of asset markets under two-tiered asymmetric information. Asset market equilibrium under ambiguity is found in section 3. Section 4 analyzes the characteristics of the equilibrium in view of the degree of ambiguity. Concluding remarks are given in section 5.

2. THE MODEL

In a two-period economy, there are two assets: a risk-free bond and a risky asset. The economy is populated by a continuum of investors, indexed in the interval $[0, 1]$. Taking the bond as the numeraire, let P be the price of the risky asset in the first period, when investor t invests his initial wealth w_t between b_t shares of the bond and x_t shares of the risky asset with the budget constraint $b_t + Px_t = w_t$. In the second period, the bond and the risky asset yield 1 and \tilde{v} , respectively, and thus his portfolio (b_t, x_t) yields wealth $w_t = w_t + (\tilde{v} - P)x_t$. The payoff \tilde{v} of the risky asset is the sum of true value $\tilde{\theta}$ and noise $\tilde{\varepsilon}$: $\tilde{v} = \tilde{\theta} + \tilde{\varepsilon}$, where $\tilde{\theta}$ and $\tilde{\varepsilon}$ are normal random variables with means μ and 0 and variances σ_θ^2 and σ_ε^2 , respectively. Random supply per capita \tilde{z} of the risky asset is also assumed to be normally distributed with mean 0 and variance σ_z^2 . All random variables are independent. All the investors have rational expectations so that they understand the functional relationship \tilde{p} between p and (θ, z) with $\tilde{p}(\theta, z) = p$. They have the same CARA utility function with the coefficient of constant absolute risk aversion $\alpha > 0$: $u(c) = -\exp(-\alpha c)$ as in Grossman and Stiglitz (1980) and Mele and Sangiorgi (2011).

As mentioned in the introduction, all the investors are divided into three groups: informed investors, uninformed investors without ambiguity (we refer to as *non-ambiguous investors*), and uninformed investors with ambiguity (we refer to as *ambiguous investors*). Informed investors

when there is ambiguous information only about volatility.

observe realization θ of $\tilde{\theta}$ with p , while uninformed investors only observe p . Non-ambiguous investors know the distribution of $\tilde{\theta}$, while ambiguous investors only know that $\mu \in [\underline{\mu}, \bar{\mu}]$ with the exact information about σ_θ^2 as in Mele and Sangiorgi (2011). Length $\Delta\mu = \bar{\mu} - \underline{\mu}$ of the interval is called *individual degree of ambiguity*. Thus, the first tier of asymmetric information about θ exists between informed investors and uninformed investors, while the second tier of asymmetric information about the distribution of θ exists between non-ambiguous investors and ambiguous investors.

All investors in each group are identical. Let $\lambda_1 \in (0, 1)$ denote the fraction of informed investors and $\lambda_2 \in [0, 1]$ that of non-ambiguous investors among uninformed investors. The proportion $(1 - \lambda_2)$ is called *market degree of ambiguity*. Note that we exclude the case where all investors are either informed or uninformed. It is assumed that λ_1 and λ_2 are exogenously given so that there is no endogenous information acquisition. Our model reduces to that of Grossman and Stiglitz (1980) when $\lambda_2 = 1$ or $\Delta\mu = 0$ and to that of Mele and Sangiorgi (2011) when $\lambda_2 = 0$.⁷⁾

For the optimal portfolio choice, informed investor i with initial wealth w_i solves

$$\max_{x_i} \mathbb{E}[-\exp(-\alpha[w_i + (\tilde{v} - p)x_i]) | (\tilde{p}, \tilde{\theta}) = (p, \theta)],$$

and his demand for the risky asset is given by

$$x_i(p, \theta) = \frac{\theta - p}{\alpha\sigma_\varepsilon^2}. \quad (1)$$

Non-ambiguous investor n with initial wealth w_n solves

⁷⁾ Here we consider the case where there is no information acquisition in their models.

$$\max_{x'_n} \mathbb{E} \left[-\exp(-\alpha[w_n + (\tilde{v} - p)x'_n]) \mid \tilde{p} = p \right],$$

and his demand for the risky asset is given by

$$x_n(p, \tilde{p}) = \frac{\mathbb{E}_\mu[\tilde{v} \mid \tilde{p} = p] - p}{\alpha \text{Var}[\tilde{v} \mid \tilde{p} = p]}. \quad (2)$$

Ambiguous investor a chooses the optimal portfolio according to the maxmin expected utility of Gilboa and Schmeidler (1989). Thus he solves

$$\max_{x'_a} \min_{\mu_a \in \{\underline{\mu}, \bar{\mu}\}} \mathbb{E}_{\mu_a} \left[-\exp(-\alpha[w_a + (\tilde{v} - p)x'_a]) \mid \tilde{p} = p \right],$$

where w_a is his initial wealth. Then his demand for the risky asset is given by

$$x_a(p, \tilde{p}) = \begin{cases} \frac{\mathbb{E}_\mu[\tilde{v} \mid \tilde{p} = p] - p}{\alpha \text{Var}[\tilde{v} \mid \tilde{p} = p]}, & \text{if } p < \mathbb{E}_\mu[\tilde{v} \mid \tilde{p} = p], \\ 0, & \text{if } \mathbb{E}_\mu[\tilde{v} \mid \tilde{p} = p] \leq p \leq \mathbb{E}_{\bar{\mu}}[\tilde{v} \mid \tilde{p} = p], \\ \frac{\mathbb{E}_{\bar{\mu}}[\tilde{v} \mid \tilde{p} = p] - p}{\alpha \text{Var}[\tilde{v} \mid \tilde{p} = p]}, & \text{if } p > \mathbb{E}_{\bar{\mu}}[\tilde{v} \mid \tilde{p} = p]. \end{cases} \quad (3)$$

It is noted that ambiguous investors participate in trading of the risky asset when its price is sufficiently low or sufficiently high for them. This means that ambiguous investors are cautious to take positions in the risky asset.

3. ASSET MARKET EQUILIBRIUM

We adopt the notion of rational expectations equilibrium in Grossman and Stiglitz (1980). By (1)-(3), the (risky) asset market is cleared at p if

$$\lambda_1 x_i(p, \theta) + (1 - \lambda_1) \lambda_2 x_n(p, \tilde{p}) + (1 - \lambda_1)(1 - \lambda_2) x_a(p, \tilde{p}) = z. \quad (4)$$

Following Grossman and Stiglitz (1980), we define a function \tilde{s} as

$$\tilde{s}(\theta, z) = \theta - \frac{\alpha \sigma_\varepsilon^2}{\lambda_1} z,$$

whose realization is denoted by $s \in S$. Note that \tilde{s} is a normal random variable with mean μ and variance $\sigma_s^2 = \sigma_\theta^2 + \alpha^2 \sigma_\varepsilon^4 \sigma_z^2 / \lambda_1^2$. The function \tilde{s} is θ plus a white noise and thus provides a partial information about θ , in which sense it is sometimes called a signal function. Let us conjecture that equilibrium asset price is represented by a function P of s such that $P(s) = P(\tilde{s}(\theta, z)) := \tilde{p}(\theta, z)$ with $\tilde{s}(\theta, z) = s$, which is verified by Theorem 1 below. For simplicity, henceforth we set $\mu = 0$ and $\underline{\mu} = -\bar{\mu}$, so that $\Delta\mu = 2\bar{\mu}$.

Now we formally define a rational expectations equilibrium.

Definition 1: A rational expectations equilibrium consists of equilibrium asset price function P and equilibrium demand function $(\tilde{x}_i, \tilde{x}_n, \tilde{x}_a)$ such that, for $P(s)$ with $s = \tilde{s}(\theta, z)$,

1. $\tilde{x}_i(s) = x_i(p, \theta)$; $\tilde{x}_n(s) = x_n(p, \tilde{p})$; $\tilde{x}_a(\theta, z) = x_a(p, \tilde{p})$;
2. $\lambda_1 \tilde{x}_i(s) + (1 - \lambda_1) \lambda_2 \tilde{x}_n(s) + (1 - \lambda_1)(1 - \lambda_2) \tilde{x}_a(s) = z$.

Then we can state our main theorem as follows.

Theorem 1: There exists a unique rational expectations equilibrium asset price function given by

$$P(s) = (\underline{\kappa} + \zeta s) 1_{(-\infty, \underline{s})}(s) + \zeta_1 s 1_{[\underline{s}, \bar{s}]}(s) + (\bar{\kappa} + \zeta s) 1_{(\bar{s}, \infty)}(s), \quad (5)$$

where $1_A(\cdot)$ is an indicator function for a set A in \mathbb{R} and

$$\zeta = \frac{\lambda_1(\lambda_1\sigma_\theta^2 + \alpha^2\sigma_\theta^2\sigma_\varepsilon^2\sigma_z^2 + \alpha^2\sigma_\varepsilon^4\sigma_z^2)}{\lambda_1^2\sigma_\theta^2 + \lambda_1\alpha^2\sigma_\theta^2\sigma_\varepsilon^2\sigma_z^2 + \alpha^2\sigma_\varepsilon^4\sigma_z^2},$$

$$\zeta_1 = \frac{\lambda_1[(1-\lambda_1)\lambda_1\lambda_2\sigma_\theta^2 + \lambda_1^2\sigma_\theta^2 + \alpha^2\sigma_\theta^2\sigma_\varepsilon^2\sigma_z^2 + \alpha^2\sigma_\varepsilon^4\sigma_z^2]}{\{(1-\lambda_1)\lambda_2 + \lambda_1\}(\lambda_1^2\sigma_\theta^2 + \alpha^2\sigma_\varepsilon^4\sigma_z^2) + \lambda_1\alpha^2\sigma_\theta^2\sigma_\varepsilon^2\sigma_z^2},$$

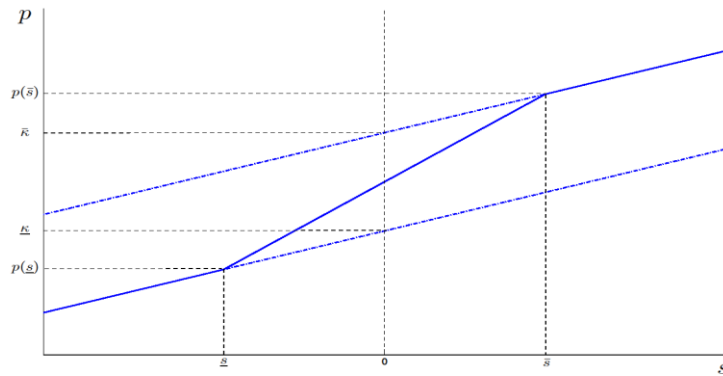
$$\bar{\kappa} = -\underline{\kappa} = \frac{(1-\lambda_1)(1-\lambda_2)\alpha^2\sigma_\varepsilon^4\sigma_z^2\Delta\mu}{2(\lambda_1^2\sigma_\theta^2 + \lambda_1\alpha^2\sigma_\theta^2\sigma_\varepsilon^2\sigma_z^2 + \alpha^2\sigma_\varepsilon^4\sigma_z^2)},$$

$$\bar{s} = -\underline{s} = \frac{[\{(1-\lambda_1)\lambda_2 + \lambda_1\}(\lambda_1^2\sigma_\theta^2 + \alpha^2\sigma_\varepsilon^4\sigma_z^2) + \lambda_1\alpha^2\sigma_\theta^2\sigma_\varepsilon^2\sigma_z^2]\Delta\mu}{2\lambda_1(\lambda_1^2\sigma_\theta^2 + \alpha^2\sigma_\theta^2\sigma_\varepsilon^2\sigma_z^2 + \alpha^2\sigma_\varepsilon^4\sigma_z^2)}.$$

Proof: See the Appendix.

The equilibrium asset price function P is piecewise linear in s as illustrated in figure 1. Since P strictly increases in s by (5), the information from observed asset price p is equivalent to that from s and therefore it gives only partial information about θ to the uninformed investors.

Figure 1 Equilibrium Price Function P
when $(\lambda_1, \Delta\mu, \alpha, \sigma_\theta^2, \sigma_\varepsilon^2, \sigma_z^2) = (0.1, 2, 1, 1, 1, 1)$



It is noted that the price function has kinks at \underline{s} and \bar{s} . This is because ambiguous investors do not trade when $s \in [\underline{s}, \bar{s}]$. Thus we call the interval $[\underline{s}, \bar{s}]$ *non-participation region* of ambiguous investors whose size is given by $\Delta s = 2\bar{s}$, which increases in individual degree of ambiguity $\Delta\mu$ and decreases in the fraction $(1-\lambda_2)$ of ambiguous investors among uninformed investors. The slope ζ_1 on non-participation region is independent from $\Delta\mu$ and increases in the proportion $(1-\lambda_2)$ of ambiguous investors among uninformed investors since

$$\frac{\partial \zeta_1}{\partial \lambda_2} = - \frac{(1-\lambda_1)\lambda_1\alpha^2\sigma_\varepsilon^4\sigma_z^2(\lambda^2\sigma_\theta^2 + \alpha^2\sigma_\theta^2\sigma_\varepsilon^2\sigma_z^2 + \alpha^2\sigma_\varepsilon^4\sigma_z^2)}{[\{(1-\lambda_1)\lambda_2 + \lambda_1\}(\lambda_1^2\sigma_\theta^2 + \alpha^2\sigma_\varepsilon^4\sigma_z^2) + \lambda_1\alpha^2\sigma_\theta^2\sigma_\varepsilon^2\sigma_z^2]^2} < 0. \quad (6)$$

Note that ζ_1 has minimum value ζ when $(1-\lambda_2)=0$ and maximum value 1 when $(1-\lambda_2)=1$ by Theorem 1 and (6). This implies that $\zeta_1 > \zeta$ as long as there exist ambiguous investors. Figure 2 and figure 3 illustrate the changes of price function P and $\Delta\mu$ or $(1-\lambda_2)$ changes, respectively.

Let $\rho \equiv (1-\lambda_2)\Delta\mu \geq 0$. We say that ambiguity is absent if there are no ambiguous investors or the individual degree of ambiguity is zero, i.e., $\rho = 0$ and that ambiguity is present otherwise.

Corollary 1: *The following hold.*

1. *If ambiguity is absent, i.e., $\rho = 0$, then the equilibrium asset price P becomes*

$$P_0(s) = \zeta s, \quad \forall s \in \mathbb{R}.$$

2. *If all the uninformed investors are ambiguous, i.e., $(1-\lambda_2)=1$, then P becomes*

$$P_1(s) = (\underline{\kappa}' + \zeta s)_{(-\infty, \underline{s})}(s) + s_{[\underline{s}, \bar{s}]}(s) + (\bar{\kappa}' + \zeta s)_{(\bar{s}, \infty)}(s), \quad \forall s \in \mathbb{R},$$

Figure 2 Changes of P as $\Delta\mu$ increases when $(\lambda_1, \lambda_2, \alpha, \sigma_\theta^2, \sigma_\varepsilon^2, \sigma_z^2) = (0.1, 0.3, 1, 1, 1, 1)$

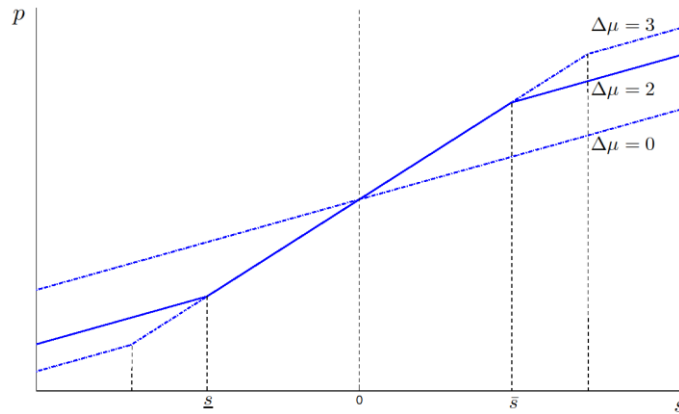
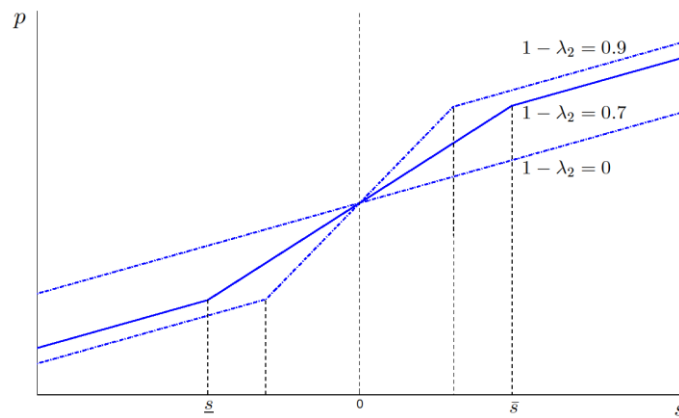


Figure 3 Changes of P as $(1 - \lambda_2)$ increases when $(\lambda_1, \Delta\mu, \alpha, \sigma_\theta^2, \sigma_\varepsilon^2, \sigma_z^2) = (0.1, 2, 1, 1, 1, 1)$



where $\bar{\kappa}' = -\underline{\kappa}' = -\frac{(1 - \lambda_1)\alpha^2\sigma_\varepsilon^4\sigma_z^2\Delta\mu}{2(\lambda_1^2\sigma_\theta^2 + \lambda_1\alpha^2\sigma_\theta^2\sigma_\varepsilon^2\sigma_z^2 + \alpha^2\sigma_\varepsilon^4\sigma_z^2)}$.

Grossman and Stiglitz (GS, 1980) and Mele and Sangiorgi (MS, 2011) correspond to (1) and (2) of Corollary 1, respectively. If $\rho = 0$, then P becomes linear. If $\rho > 0$ and $(1 - \lambda_2) = 1$, then P exhibits maximum slope 1 on the non-participation region.

4. EFFECTS OF AMBIGUITY ON ASSET MARKET EQUILIBRIUM

Under two-tiered asymmetric information, we examine effects of ambiguity on liquidity risk, price sensitivity, and volatility in view of both the individual or market degree of ambiguity. For simplicity, we fix the proportions of informed and uninformed investors throughout this section. Thus comparative statics will be done by changing the individual degree $\Delta\mu$ of ambiguity and the proportion $(1 - \lambda_2)$ of ambiguous investors among the uninformed, which is called market degree of ambiguity.

Now let us define

$$\eta = \frac{\bar{s}}{\sqrt{2\sigma_s^2}}, \quad \text{erf}(\eta) = \frac{2}{\sqrt{\pi}} \int_0^\eta \exp(-t^2) dt, \quad \text{and} \quad f_s(s) = \frac{1}{\sqrt{2\pi\sigma_s^2}} \exp\left(-\frac{s^2}{2\sigma_s^2}\right).$$

Note that $\text{erf}(\eta) = 2 \int_0^{\bar{s}} f_s(s) ds$, which is the probability that an ambiguous investor does not trade.

4.1. Liquidity Risk

In Ozsoylev and Werner (2011), liquidity risk is defined as the probability that asset price lies in the non-participation region of ambiguous investors. In our model, the liquidity risk is defined by the probability that an investor does not trade the risky asset, i.e.,

$$L = (1 - \lambda_1)(1 - \lambda_2) \int_s^{\bar{s}} f_s(s) ds = (1 - \lambda_1)(1 - \lambda_2) \text{erf}(\eta).$$

This consists of two parts: the population of ambiguous investors and the probability that an ambiguous investor does not trade. In particular, $L = 0$ if $\rho = 0$. When $\rho > 0$, the next proposition shows how ambiguity affects liquidity risk. Let us consider a strictly increasing function of $\Delta\mu > 0$, which is given by

$$h(\Delta\mu) \equiv \sqrt{\pi} \text{erf} \left(\frac{\Delta\mu}{2\sqrt{2}\sigma_s} \right) \left[\Delta\mu \exp \left(-\frac{(\Delta\mu)^2}{8\sigma_s^2} \right) \right]^{-1}.$$

Proposition 1: *If $\rho > 0$, then the following hold.*

1. *Liquidity risk L increases in the individual degree $\Delta\mu$ of ambiguity.*
2. *L increases in the fraction $(1 - \lambda_2)$ of ambiguous investors among uninformed investors if $\Delta\mu$ is sufficiently large such that $h(\Delta\mu) \geq 1/\lambda_1 - 1$.*

Proof: (1) It is straightforward since $\text{erf}(\eta)$ increases in $\Delta\mu$.

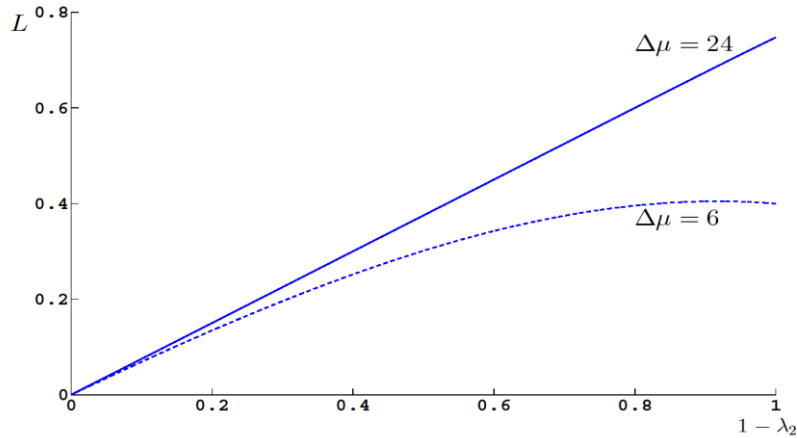
(2) Since $\partial\eta/\partial\lambda_2 < (1/\lambda_1 - 1)\Delta\mu/2$ and $h(\Delta\mu) \geq 1/\lambda_1 - 1$, we have

$$\begin{aligned} \left. \frac{\partial L}{\partial \lambda_2} \right|_{\lambda_2=0} &= -(1 - \lambda_1) \left[\text{erf} \left(\frac{\Delta\mu}{2\sqrt{2}\sigma_s} \right) - \frac{2}{\sqrt{\pi}} \exp \left(-\frac{(\Delta\mu)^2}{8\sigma_s^2} \right) \frac{\partial \eta}{\partial \lambda_2} \right] \\ &< -\frac{(1 - \lambda_1)\Delta\mu}{\sqrt{\pi}} \exp \left(-\frac{(\Delta\mu)^2}{8\sigma_s^2} \right) \left[h(\Delta\mu) - \left(\frac{1}{\lambda_1} - 1 \right) \right] \\ &\leq 0. \end{aligned}$$

Moreover, we also have $\partial^2 L / \partial \lambda_2^2 < 0$ for all $\lambda_2 \in [0, 1]$.

Consequently, it follows that $\partial L / \partial \lambda_2 < 0$ for all $\lambda_2 \in [0, 1]$, which implies the claim.

Figure 4 Changes of Liquidity Risk L for $(1-\lambda_2)$
when $(\lambda_1, \alpha, \sigma_\theta^2, \sigma_\varepsilon^2, \sigma_z^2) = (0.25, 1, 1, 1, 1)$



If the individual degree $\Delta\mu$ of ambiguity increases, the size Δs of non-participation region increases by Theorem 1, and so does the probability of non-participation, which implies that liquidity risk increases. When $(1-\lambda_2)$ increases, the population of ambiguous investors increases but the size Δs of non-participation region decreases by Theorem 1, and so does the probability of non-participation. However, if $\Delta\mu$ is sufficiently large, it turns out that the first effect dominates the second, so as to increase liquidity risk, which is illustrated in the left panel of figure 4. The solid line shows that L monotonically increases when $\Delta\mu = 24$. Otherwise, liquidity risk may increase, achieve the maximum, and then decrease in $(1-\lambda_2)$, as illustrated in the right panel of figure 4. The dashed line shows L initially increases, and then decreases when $\Delta\mu = 6$.

4.2. Price Sensitivity

One unit change of asset supply z moves the asset price by $\alpha\sigma_\varepsilon^2\zeta/\lambda_1$ on $[\underline{s}, \bar{s}]^c$ and by $\alpha\sigma_\varepsilon^2\zeta_1/\lambda_1$ on $[\underline{s}, \bar{s}]$. Thus we can define price sensitivity to asset supply z as

$$\psi(s) = \frac{\alpha\sigma_\varepsilon^2}{\lambda_1} \left[\zeta_{[\underline{s}, \bar{s}]^c}(s) + \zeta_{I[\underline{s}, \bar{s}]}(s) \right].$$

In particular, the price sensitivity in Grossman and Stiglitz (1980) is given by $\alpha\sigma_\varepsilon^2\zeta / \lambda_1$. The presence of ambiguity leads to the relation $\zeta_1 > \zeta$, implying that asset price is more sensitive on $[\underline{s}, \bar{s}]$ than on $[\underline{s}, \bar{s}]^c$. Since the price sensitivity depends on s , it seems to be appropriate to use *expected price sensitivity* $\mathbb{E}[\psi]$ in analyzing the overall effect of ambiguity on asset market equilibrium.

Proposition 2: *If $\rho > 0$, then the following hold.*

1. *Expected price sensitivity $\mathbb{E}[\psi]$ is greater under ambiguity than under no ambiguity by $\alpha\sigma_\varepsilon^2(\zeta_1 - \zeta)\text{erf}(\eta) / \lambda_1 > 0$.*
2. *$\mathbb{E}[\psi]$ increases in the individual degree $\Delta\mu$ of ambiguity.*
3. *$\mathbb{E}[\psi]$ increases in the fraction $(1 - \lambda_2)$ of ambiguous investors among uninformed investors.*

Proof: (1) It is obvious that $\mathbb{E}[\psi] = \alpha\sigma_\varepsilon^2\zeta / \lambda_1$ when $\rho = 0$. If $\rho > 0$, we have

$$\mathbb{E}[\psi] = \frac{\alpha\sigma_\varepsilon^2}{\lambda_1} \left[\zeta + (\zeta_1 - \zeta) \int_{\underline{s}}^{\bar{s}} f_s(s) ds \right] = \frac{\alpha\sigma_\varepsilon^2}{\lambda_1} \zeta + \frac{\alpha\sigma_\varepsilon^2}{\lambda_1} (\zeta_1 - \zeta) \text{erf}(\eta).$$

Since $\zeta_1 > \zeta$ and $\eta > 0$, it holds that , which implies the claim.

(2) It holds since $\text{erf}(\eta)$ increases in $\Delta\mu$.

(3) The partial derivative of $\mathbb{E}[\psi]$ with respect to λ_2 is given by

$$\frac{\partial \mathbb{E}[\psi]}{\partial \lambda_2} = \frac{\alpha\sigma_\varepsilon^2}{\lambda_1} \left[\frac{\partial \zeta_1}{\partial \lambda_2} \text{erf}(\eta) + \frac{2(\zeta_1 - \zeta)}{\sqrt{\pi}} \exp(-\eta^2) \frac{\partial \eta}{\partial \lambda_2} \right].$$

Let $g(\Delta\mu) \equiv \partial \mathbb{E}[\psi] / \partial \lambda_2$. Since $\text{erf}(\eta) = \partial \eta / \partial \lambda_2 = 0$ if $\Delta\mu = 0$, we have $g(0) = 0$ and, for all $\Delta\mu > 0$,

$$g'(\Delta\mu) = -\frac{2\alpha\sigma_\varepsilon^2}{\sqrt{\pi}\lambda_1} \exp(-\eta^2)$$

$$\left[\frac{(1-\lambda)\alpha^2\sigma_\varepsilon^4\sigma_z^2}{\sqrt{2\sigma_s^2(\lambda_1^2\sigma_\theta^2 + \lambda\alpha^2\sigma_\theta^2\sigma_\varepsilon^2\sigma_z^2 + \alpha^2\sigma_\varepsilon^4\sigma_z^2)}} + 2(\zeta_1 - \zeta)\eta \frac{\partial\eta}{\partial(\Delta\mu)} \right] < 0.$$

Thus $g(\Delta\mu) < 0$ for all $\Delta\mu > 0$, so that $\partial\mathbb{E}[\psi]/\partial\lambda_2 < 0$. Hence the claim follows.

Intuitively, (2) of Proposition 2 follows from the fact that an increase of $\Delta\mu$ widens the non-participation region holding the corresponding slope $\alpha\sigma_\varepsilon^2\zeta_1/\lambda_1$ constant as illustrated in figure 2. If $(1-\lambda_2)$ increases, the size Δs of non-participation region decreases, while $\alpha\sigma_\varepsilon^2\zeta_1/\lambda_1$ on $[\underline{s}, \bar{s}]$ increases by (6), which is illustrated in figure 3. However, from (3) of Proposition 2, we see that the latter effect dominates the former one.

A closely related notion with price sensitivity is market depth. The asset market is said to be deep when the price can absorb random supply shock without much variation. Kyle (1985) measures market depth by the inverse of price sensitivity. In our model, *market depth* can be defined by the reciprocal of the expected price sensitivity. Hence, Proposition 2 implies that market depth is greater under no ambiguity than under ambiguity. Furthermore, the market depth decreases in $\Delta\mu$ and $(1-\lambda_2)$. These results are summarized in Proposition 3.

Proposition 3: *If $\rho > 0$, then the following hold.*

1. *Market depth $1/\mathbb{E}[\psi]$ is smaller under ambiguity than under no ambiguity.*
2. *$1/\mathbb{E}[\psi]$ decreases in the individual degree $\Delta\mu$ of ambiguity.*
3. *$1/\mathbb{E}[\psi]$ decreases in the fraction $(1-\lambda_2)$ of ambiguous investors among uninformed investors.*

4.3. Price Volatility

From the results for expected price sensitivity of Proposition 2, one can expect that price volatility σ_p^2 is greater under ambiguity than under no ambiguity and moreover increases as the ‘degree of ambiguity’ increases. The following proposition verifies that this is true.

Proposition 4: *If $\rho > 0$, then the following hold.*

1. *Price volatility σ_p^2 is greater under ambiguity than under no ambiguity by 2ξ where*

$$\begin{aligned} \xi = & \frac{\bar{\kappa}^2}{2}(1 - \text{erf}(\eta)) + \sqrt{\frac{2}{\pi}}\bar{\kappa}\zeta\sigma_s \exp(-\eta^2) \\ & + (\zeta_1^2 - \zeta^2)\sigma_s^2 \left(\frac{1}{2}\text{erf}(\eta) - \frac{\eta}{\sqrt{\pi}}\exp(-\eta^2) \right) > 0. \end{aligned}$$

2. σ_p^2 increases in the individual degree $\Delta\mu$ of ambiguity.
3. σ_p^2 increases in the fraction $(1 - \lambda_2)$ of ambiguous investors among uninformed investors.

Proof: (1) Recalling $\mathbb{E}[P] = 0$, we see

$$\begin{aligned} \sigma_p^2 &= \zeta^2\sigma_s^2 + 2 \left[\bar{\kappa}^2 \int_{\bar{s}}^{\infty} f_s(s)ds + 2\bar{\kappa}\zeta \int_{\bar{s}}^{\infty} sf_s(s)ds + (\zeta_1^2 - \zeta^2) \int_0^{\bar{s}} s^2 f_s(s)ds \right] \\ &= \zeta^2\sigma_s^2 + 2 \left[\frac{\bar{\kappa}^2}{2}(1 - \text{erf}(\eta)) + \sqrt{\frac{2}{\pi}}\bar{\kappa}\zeta\sigma_s \exp(-\eta^2) \right. \\ & \quad \left. + (\zeta_1^2 - \zeta^2)\sigma_s^2 \left(\frac{1}{2}\text{erf}(\eta) - \frac{\eta}{\sqrt{\pi}}\exp(-\eta^2) \right) \right] \\ &= \sigma_{p_0}^2 + 2\xi, \end{aligned}$$

where $\sigma_{p_0}^2$ is price volatility when $\rho = 0$. Since $\xi > 0$ if and only if $\rho > 0$, we have $\sigma_p^2 > \sigma_{p_0}^2$.

(2) Noting that $\zeta_1 \bar{s} = \bar{\kappa} + \zeta \bar{s}$, we have

$$\frac{\partial \sigma_p^2}{\partial (\Delta\mu)} = 2 \left[\bar{\kappa}(1 - \text{erf}(\eta)) + \sqrt{\frac{2}{\pi}} \zeta \sigma_s \exp(-\eta^2) \right] \frac{\partial \bar{\kappa}}{\partial \Delta\mu} > 0,$$

which implies the claim.

(3) Similarly, since

$$\begin{aligned} \frac{\partial \sigma_p^2}{\partial \lambda_2} &= 2 \left[\bar{\kappa}(1 - \text{erf}(\eta)) + \sqrt{\frac{2}{\pi}} \zeta \sigma_s \exp(-\eta^2) \right] \frac{\partial \bar{\kappa}}{\partial \lambda_2} \\ &\quad + 4\zeta_1 \sigma_s^2 \left[\frac{1}{2} \text{erf}(\eta) - \frac{\eta}{\sqrt{\pi}} \exp(-\eta^2) \right] \frac{\partial \zeta_1}{\partial \lambda_2} < 0, \end{aligned}$$

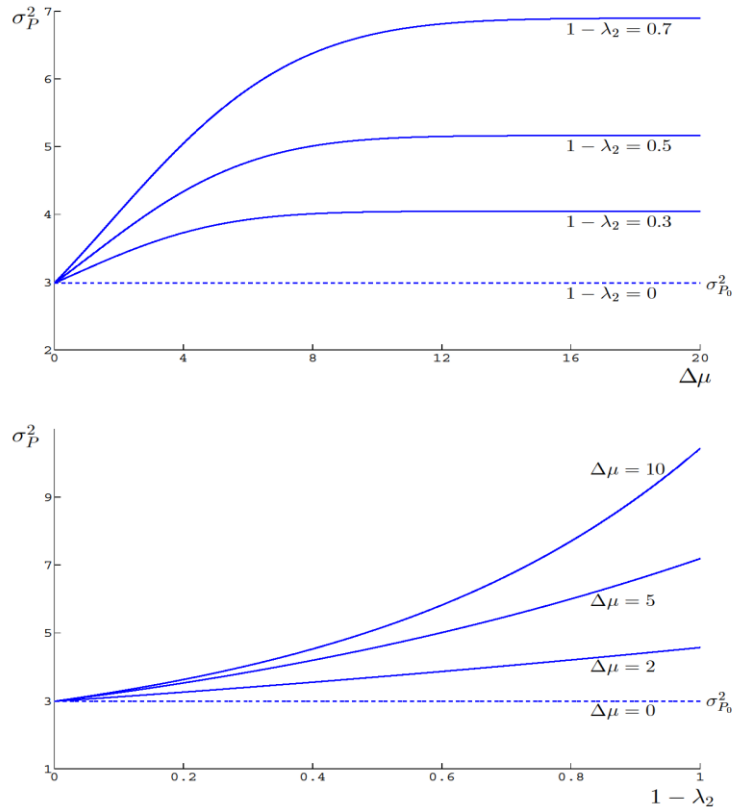
the claim holds.

Changes of price volatility σ_p^2 with respect to $\Delta\mu$ and $(1-\lambda_2)$ are illustrated in figure 5. In the presence of ambiguity, σ_p^2 is greater than that in the absence of ambiguity (depicted in dashed lines). As the individual degree $\Delta\mu$ of ambiguity increases, so does the probability that s falls in non-participation region $[\underline{s}, \bar{s}]$ where the slope of P is steeper, which implies σ_p^2 increases. It is noted that σ_p^2 increases in concave fashion as shown in the upper panel of figure 5. Interestingly, σ_p^2 converges to the supremum $\zeta_1^2 \sigma_s^2$ even when $\Delta\mu$ diverges to infinity.⁸⁾ It is because price function P becomes linear in s with slope ζ_1 . Furthermore, price volatility becomes quite close to $\zeta_1^2 \sigma_s^2$ when sufficiently large $\Delta\mu$.

The increase of $(1-\lambda_2)$ will raise the slope ζ_1 on non-participation region, pushing up σ_p^2 . On the other hand, it will reduce the size of non-participation region, pushing down σ_p^2 . It turns out that the first effect dominates the second one and thus σ_p^2 increases. The lower panel of figure 5 shows that σ_p^2 increases in $(1-\lambda_2)$ in convex fashion.

⁸⁾ In this case, all the ambiguous investors do not participate in the asset markets for any observed price.

Figure 5 Changes of Price volatility σ_P^2 for $\Delta\mu$ and $(1-\lambda_2)$ when $(\lambda_1, \alpha, \sigma_\theta^2, \sigma_\varepsilon^2, \sigma_z^2) = (0.3, 1, 1, 1, 1)$



5. CONCLUSION

Introducing ambiguous investors into Grossman and Stiglitz’s (1980) model, the paper analyzes equilibrium asset price under the two-tiered asymmetric information. If the individual or market degree of ambiguity increases, then liquidity risk, expected price sensitivity, and price volatility would increase while market depth decreases. These properties of the

equilibrium price are brought by the non-participation of ambiguous investors in some price region. Obviously, an important direction for future research is to incorporate endogenous information acquisition into our model.

APPENDIX

Proof of Theorem 3.1: Suppose $p < \mathbb{E}_{\underline{\mu}}[\tilde{v} | P = p]$. We assume that P is a linear function of s such that $P(s) = \underline{\kappa} + \zeta s$. Then the information from p becomes equivalent to that from s and hence we have

$$\mathbb{E}_{\underline{\mu}}[\tilde{v} | P = p] = \mathbb{E}_{\underline{\mu}}[\tilde{v} | \tilde{s} = s] = \frac{\alpha^2 \sigma_{\varepsilon}^4 \sigma_z^2 \underline{\mu} + \lambda_1^2 \sigma_{\theta}^2 s}{\lambda_1^2 \sigma_{\theta}^2 + \alpha^2 \sigma_{\varepsilon}^4 \sigma_z^2},$$

$$\text{Var}[\tilde{v} | P = p] = \text{Var}[\tilde{v} | \tilde{s} = s] = \frac{\sigma_{\varepsilon}^2 (\lambda_1^2 \sigma_{\theta}^2 + \alpha^2 \sigma_{\theta}^2 \sigma_{\varepsilon}^2 \sigma_z^2 + \alpha^2 \sigma_{\varepsilon}^4 \sigma_z^2)}{\lambda_1^2 \sigma_{\theta}^2 + \alpha^2 \sigma_{\varepsilon}^4 \sigma_z^2}.$$

From (1)-(4), we obtain

$$\begin{aligned} P(s) &= \frac{(1 - \lambda_1)(1 - \lambda_2)\sigma_{\varepsilon}^2 \mathbb{E}_{\underline{\mu}}[\tilde{v} | \tilde{s} = s] + (1 - \lambda_1)\lambda_2\sigma_{\varepsilon}^2 \mathbb{E}[\tilde{v} | \tilde{s} = s] + \lambda_1 s \text{Var}[\tilde{v} | \tilde{s} = s]}{(1 - \lambda_1)(1 - \lambda_2)\sigma_{\varepsilon}^2 + (1 - \lambda_1)\lambda_2\sigma_{\varepsilon}^2 + \lambda_1 \text{Var}[\tilde{v} | \tilde{s} = s]} \\ &= -\frac{(1 - \lambda_1)(1 - \lambda_2)\alpha^2 \sigma_{\varepsilon}^4 \sigma_z^2 \Delta \mu}{2(\lambda_1^2 \sigma_{\theta}^2 + \lambda_1 \alpha^2 \sigma_{\theta}^2 \sigma_{\varepsilon}^2 \sigma_z^2 + \alpha^2 \sigma_{\varepsilon}^4 \sigma_z^2)} \\ &\quad + \frac{\lambda_1 (\lambda_1 \sigma_{\theta}^2 + \alpha^2 \sigma_{\theta}^2 \sigma_{\varepsilon}^2 \sigma_z^2 + \alpha^2 \sigma_{\varepsilon}^4 \sigma_z^2)}{\lambda_1^2 \sigma_{\theta}^2 + \lambda_1 \alpha^2 \sigma_{\theta}^2 \sigma_{\varepsilon}^2 \sigma_z^2 + \alpha^2 \sigma_{\varepsilon}^4 \sigma_z^2} s. \end{aligned}$$

Similarly, equilibrium asset price function for $p > \mathbb{E}_{\underline{\mu}}[\tilde{v} | P = p]$ is given by

$$P(s) = \frac{(1-\lambda_1)(1-\lambda_2)\alpha^2\sigma_\varepsilon^4\sigma_z^2\Delta\mu}{2(\lambda_1^2\sigma_\theta^2 + \lambda_1\alpha^2\sigma_\theta^2\sigma_\varepsilon^2\sigma_z^2 + \alpha^2\sigma_\varepsilon^4\sigma_z^2)} + \frac{\lambda_1(\lambda_1\sigma_\theta^2 + \alpha^2\sigma_\theta^2\sigma_\varepsilon^2\sigma_z^2 + \alpha^2\sigma_\varepsilon^4\sigma_z^2)}{\lambda_1^2\sigma_\theta^2 + \lambda_1\alpha^2\sigma_\theta^2\sigma_\varepsilon^2\sigma_z^2 + \alpha^2\sigma_\varepsilon^4\sigma_z^2} s,$$

and, when $\mathbb{E}_\mu[\tilde{v} | P = p] \leq p \leq \mathbb{E}_\mu[\tilde{v} | P = p]$, it is given by

$$P(s) = \frac{\lambda_1[(1-\lambda_1)\lambda_1\lambda_2\sigma_\theta^2 + \lambda_1^2\sigma_\theta^2 + \alpha^2\sigma_\theta^2\sigma_\varepsilon^2\sigma_z^2 + \alpha^2\sigma_\varepsilon^4\sigma_z^2]}{\{(1-\lambda_1)\lambda_2 + \lambda_1\}(\lambda_1^2\sigma_\theta^2 + \alpha^2\sigma_\varepsilon^4\sigma_z^2) + \lambda_1\alpha^2\sigma_\theta^2\sigma_\varepsilon^2\sigma_z^2} s.$$

Breaking points \underline{s} and \bar{s} are obtained by solving $\mathbb{E}_\mu[\tilde{v} | \tilde{s} = \underline{s}] = P(\underline{s})$ and $\mathbb{E}_\mu[\tilde{v} | \tilde{s} = \bar{s}] = P(\bar{s})$. Then we have

$$\bar{s} = -\underline{s} = \frac{[\{(1-\lambda_1)\lambda_2 + \lambda_1\}(\lambda_1^2\sigma_\theta^2 + \alpha^2\sigma_\varepsilon^4\sigma_z^2) + \lambda_1\alpha^2\sigma_\theta^2\sigma_\varepsilon^2\sigma_z^2]\Delta\mu}{2\lambda_1(\lambda_1^2\sigma_\theta^2 + \alpha^2\sigma_\theta^2\sigma_\varepsilon^2\sigma_z^2 + \alpha^2\sigma_\varepsilon^4\sigma_z^2)}.$$

Since $p < \mathbb{E}_\mu[\tilde{v} | P = p]$ if and only if $s < \underline{s}$ and $p > \mathbb{E}_\mu[\tilde{v} | P = p]$ if and only if $s > \bar{s}$, we obtain P as in (5).

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