

On the Global Properties of the Model of Developing Countries with Imperfect International Financial Market*

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Using a dynasty model of a representative consumer facing upward sloping supply curve of foreign debt, the characteristics of the dynamic paths of capital and consumption are examined both when the international borrowing interest rate increases with foreign debt, and with foreign-debt/capital ratio. The possibilities of bankruptcy and movement toward stationary state are examined.

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1. INTRODUCTION

This paper tries to show the global dynamic characteristics of the small country open economy facing imperfect international financial market.

The earlier work starts from Stiglitz (1970) employing Uzawa (1968a, 1968b)'s growth model, discusses the two country, two good and two factor economy and the long-run patterns of specialization. This is followed by Manning (1981).

What we are concerned with here is a small country who runs trade deficit and borrows from abroad to finance the debt by issuing bond (credit) in the international capital market with borrowing interest becoming higher, the more borrowing. The higher interest reflects the characteristics of imperfect international credit market.

Hamada (1966) discusses first the case of constant borrowing interest rate and then the case of increasing interest rate with the increase in debt, employing small country optimizing growth model.

Relating to the small country assumption, the constant interest rate requires the time discount rate to be equal to the constant interest rate at the stationary state. To avoid this rather unrealistic assumption, first adjustment costs of investment are introduced, e.g., by Sen and Turnovsky (1989), Turnovsky and Sen (1991), Turnovsky (1996) and Turnovsky (1998).

The other way to overcome this difficulty is the introduction of recursive preference as tried by e.g., Obstfeld (1982) and Karayalcin (1994).¹⁾

The significance of foreign debt to the growth of the developing countries has been surveyed by McDonald (1982) as growth-cum-debt models when the importance of foreign debt for financing investment and smoothing consumption overtime are stressed.

Relating to the imperfectness of the international capital markets, as Edwards (1984)'s empirical work suggests, the more indebted developing countries tend to pay a premium on their loans from these markets. Reflecting this empirical results, many authors assume that the borrowing

¹⁾ Karayalcin (1994) also employs the adjustment costs of investment.

interest increases as the increase in debt or foreign capital. Bardhan (1967) is the first who assumes that borrowing interest rate r increases as the increase in foreign capital, k_f . Bhandari *et al.* (1990) assumes r increases as the increase in foreign debt b , which was followed by Fisher (1995).

Another way to reflect the imperfectness of the international capital market is to assume that borrowing interest r increases as the increase in debt/capital ratio, for the default risk increases as this ratio increases and therefore the lender charges higher interest rate. This approach was adopted by, e.g., Chatterjee *et al.* (2003) and Chatterjee and Turnovsky (2005).

We follow the conventional framework of a small country open economy; there exists one good used either for consumption and investment. Production function is neoclassical with two primary factors of capital and labor, the home country borrows from abroad to finance current account deficit with borrowing interest rate r which increases as the increase in foreign debt or in foreign debt/capital ratio. Free trade of good is assumed. Factor markets are perfect domestically. We adopt so called the dynasty model, which means that the representative home consumer maximizes the discounted sum of instantaneous felicity function subject to the law of motion of capital and flow budget constraint, as shown later in more detail.

Although already many predecessors have adopted the similar model as ours and analyzed the dynamic characteristics of the optimal path of per capita consumption which depend on per capita capital stock, they are concerned with its local properties. However the truly interesting results are derived from not local but global analysis. To our best knowledge, no one has ever attempted such investigation. Our findings are in the following.

In the next section 2 we deal with the case where the borrowing interest rate increases with foreign debt. Here we show that if the stationary state exists and the initial per capita capital is above certain level, then the economy is globally stable in the sense that per capita capital and per capita consumption converges monotonically to the stationary state.

Otherwise, i.e., if the initial per capita capital is below certain level, the economy goes bankrupt even if the stable state exists, or there exists no

optimal path if there exists no stationary state. This shows the dichotomous properties of the optimal path, depending on the level of initial per capita capital k_0 . Of course these results are obtained only from the global analysis.

In section 3, we deal with the case where the borrowing interest r goes up with the increase in debt/capital ratio, and obtain the following different conclusions depending on the combinations of parameters when the stationary state exists,

(i) If the initial per capita capital k_0 is above certain level then the economy converges monotonically to the stationary state, but if k_0 is below the level then the economy goes bankrupt (Case-1). Again such dichotomous properties pertain only to the global analysis

(ii) The economy is globally stable irrespective of the level of k_0 . Section 4 concludes.

2. BORROWING INTEREST RATE DEPENDS ON DEBT

First we consider the case where the borrowing international interest rate r increases as the increase in per capita foreign debt b , i.e.,

$$r = r(b) \text{ with } r'(b) > 0 \text{ and } r''(b) > 0.$$

Here it is assumed that the rate of increase in interest rate $r'(b)$ increases as foreign debt increases.

The law of motion of domestic per capita capital k is expressed as

$$\dot{k} = i, \tag{1}$$

where \dot{k} is the time rate of change in k (in general \dot{x} is the time rate of change in variable x .) and i is the per capita investment. To finance trade

deficit the home country issues the bond b . It obeys the following law of motion

$$\dot{b} = r \cdot b - ex, \quad (2)$$

where ex is the per capita export, i.e.,

$$ex = f(k) - c - i. \quad (3)$$

Here $y = f(x)$ is the labor productivity function, y is the labor productivity, and c is the per capita consumption. The function f is neoclassical and satisfies the Inada Condition, i.e., $f' > 0$, $f'' < 0$, $\lim_{k \rightarrow 0} f'(k) = \infty$ and $\lim_{k \rightarrow 0} f'(k) = 0$. We treat the case where the home country is a debtor, i.e., $b > 0$.

Let $a = k - b$ be the per capita net wealth which is per capita capital k less per capita debt b . Then from (1), (2) and (3), we obtain the law of motion of a , i.e.,

$$\dot{a} = f(k) - c - r \cdot (k - a). \quad (4)$$

Let $u(c)$ be the felicity function of the representative consumer

$$u(c) = \frac{1}{1 - \sigma} c^{1 - \sigma},$$

where $0 < \sigma \neq 1$.²⁾

Then according to the conventional formation of the dynasty model, the representative consumer maximizes

$$\int_0^{\infty} u(c) e^{-\rho t} dt,$$

²⁾ The following arguments go through also with $u(c) = \log c$ when $\sigma = 1$.

subject to (4) where $\rho > 0$ is the time discount rate, $t = 0$ is the initial time. The consumer takes interest rate r as given although it is dependent on the debt b , since the consumer as microeconomic agent cannot control interest rate, which is determined through macroeconomic conditions. Let $W(k_0) = \max_c \int_0^\infty u(c)e^{-\rho t} dt$ be the value function which is the maximized value of the objective function subject to the law of motion of the net wealth a where k_0 is the initial value of per capita capital k .

Then the above maximization problem is solved by forming the following current value Hamiltonian

$$H = u(c) + \lambda(f(k) - c - r \cdot (k - a)), \quad (5)$$

and obtaining the first order conditions

$$c^{-\sigma} = \lambda, \quad (6)$$

$$f'(k) = r(b), \quad (7)$$

$$\dot{\lambda} = \rho\lambda - r\lambda, \quad (8)$$

and the transversality condition (TVC)

$$\lim_{t \rightarrow \infty} \lambda a e^{-\rho t} = 0. \quad (9)$$

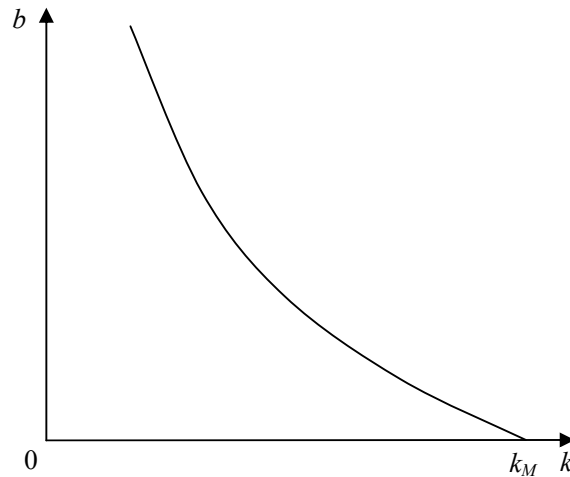
From (6), (7) and (8), we obtain

$$\dot{\lambda} = -\sigma^{-1}c(\rho - f'(k)). \quad (10)$$

From (7) we obtain $db/dk = f''/r' < 0$.

Figure 1 shows the function $b = b(k)$. Here at $k = k_M$, $f'(k_M) = r(0) = r_0$ holds. That is, k_M is the maximum value of k where the marginal of product of capital equals the world borrowing interest rate $r(0) = r_0$ which

Figure 1



would prevail if there exists no foreign borrowing.

As for net wealth $a = k - b$, we observe $da/dk = 1 - db/dk > 1$, and the amount of debt payment $g = g(k) = r \cdot b$, it follows that

$$g'(k) = r' \cdot b \cdot db/dk + r \cdot db/dk < 0,$$

with $g(k_M) = 0$. In short we obtain the following figure 2 of $f(k)$ and $g(k)$ curves. As seen in figure 2, let $k = \underline{k}$ where $g(k) = f(k)$ holds. Then $f(k) > g(k)$ if and only if $\underline{k} < k$.

Recalling $\dot{c} = f(k) - c - g(k)$, $\dot{c} > 0$ if and only if $k > \underline{k}$ from $da/dk > 1 > 0$, and from (10), we obtain the phase diagram of (k, c) for this case. There exist two subcases depending on the existence of the stationary point E . In figure 3, $\dot{k} = 0$ curve is equal to $\dot{c} = 0$ curve. That is $\dot{k} = 0$ if and only if $c = f(k) - g(k)$. Then at $k = \underline{k}$, $\dot{k} = 0$ curve intersects with the horizontal axis. From (10) with $\dot{c} = 0$, let $k = \bar{k}$ be the value of k where $f'(k) = \rho$. Then $\dot{k} = 0$ curve is the upward sloping curve and $\dot{c} = 0$ curve is the vertical line with $k = \bar{k}$. Furthermore $\dot{k} > 0$ if and only

Figure 2

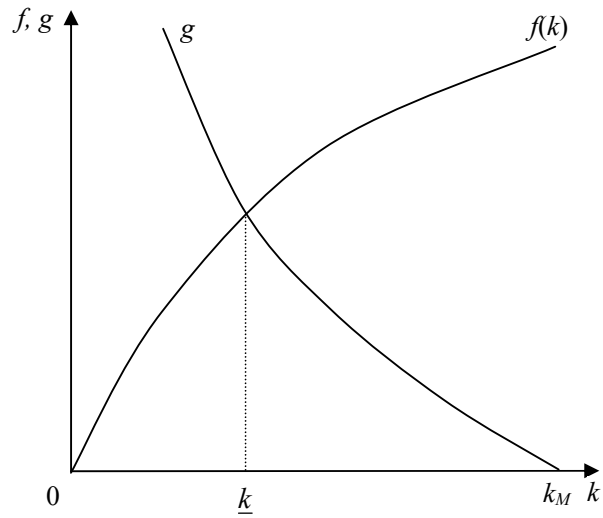
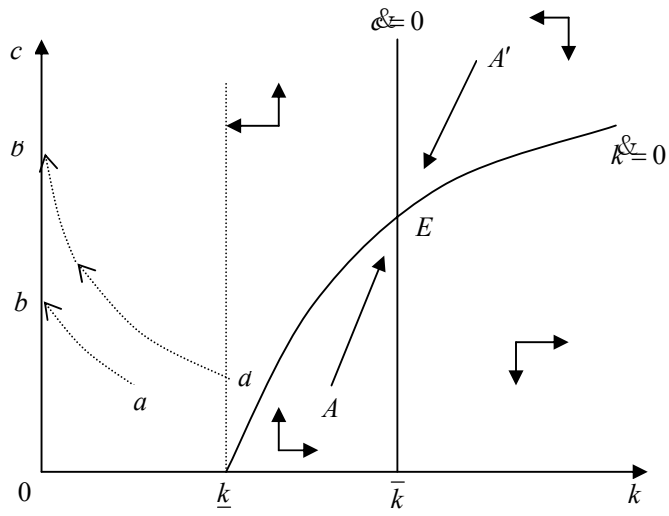


Figure 3 Case 1



if (k, c) is in the lower-right of $\dot{k}=0$ curve, and $\dot{c}>0$ if and only if (k, c) is on the left-hand-side of $\dot{c}=0$ vertical line.

First we consider Case 1 (figure 3).

Where $\underline{k} < \bar{k}$ so that the stationary point E exists.

As seen from the phase diagram (figure 3), the stationary state E is characterized by $\dot{c}=0$, i.e., $\rho = f'(k)$ and $\dot{k}=0$, i.e., $c = \bar{c} = f(\bar{k}) - g(\bar{k}) > 0$ where $\rho = f'(\bar{k})$. As seen, if $\underline{k} < k_0 \leq \bar{k}$, i.e., if the initial per capita capital k_0 is larger than \underline{k} but less than \bar{k} then the economy converges monotonically to the stationary state $E(\bar{k}, \bar{c})$ with both k and c are increasing. Similarly if $\bar{k} \leq k_0 (\leq k_M)$ then again the economy converges monotonically to the stationary state $E(\bar{k}, \bar{c})$ with both k and c being decreasing.

However if $k_0 \leq \underline{k}$, then either k becomes zero and c becomes finite in a finite time $T < \infty$ as drawn by the curve ab , or k becomes zero and c becomes infinite as $t \rightarrow \infty$ as shown by $a'b'$. For both cases TVC is seen to be violated as discussed in the following. If the optimal path of (k, c) follows ab , then it arrives at b in a finite time $t = T < \infty$. Then the transversality condition (TVC) for this kind of free finite terminal time is that the current value Hamiltonian H must vanish at $t = T$,³⁾ i.e.,

$$H = 0 \text{ at } t = T. \tag{11}$$

However as seen from (7) $k \rightarrow 0$ if and only if $f' \rightarrow \infty$ if and only if $r \rightarrow +\infty$ and hence $k \rightarrow 0$ implies $g(k) = r \cdot b \rightarrow +\infty$. Then in view of (5) with $H=0$, $\lambda = 0$ must hold at $t = T$, and hence from (6), $c \rightarrow \infty$ as $t = T$, a contradiction. This implies such path as ab violates TVC.

Next we also show such path as $a'b'$ also violates TVC (9). Here we define NPG (No-Ponzi-Game) condition which is obtained from TVC (9),

³⁾ See, e.g., Leonard and Vang Long (1992), 7.5.

$$\text{NPG: } \lim_{t \rightarrow \infty} b(t) e^{-\int_0^t r(\tau) d\tau} = 0. \quad (12)$$

However along $a'b'$, NPG is seen to be violated.⁵⁾ Intuitively NPG condition implies that the country cannot borrow more and more from abroad for financing foreign debt. Violation of NPG shows that along $a'b'$, the home country goes into bankruptcy.

Although the detailed explanation is skipped, when the initial point (k_0, c_0) is above the optimal path of (k, c) , then as seen from figure 3, along this solution path $c \rightarrow \infty$ as $k \rightarrow 0$ and $t \rightarrow \infty$ holds eventually as far as $k_0 > \underline{k}$. Then by employing exactly the same arguments as in footnote 5, this solution path is seen to violate TVC (12).

In short even if the stationary state E exists, if the initial per capita capital k is less than or equal to \underline{k} , then the economy cannot move toward the stationary state. Instead the home country goes into bankruptcy.

Next we consider Case 2 (figure 4).

Where $\bar{k} \leq \underline{k}$. For this case there exists no stationary state. Henceforth we assume that once $k = k_M$ is reached then the home country lends abroad by receiving interest rate $r(0)$. This implies that k never exceeds k_M .

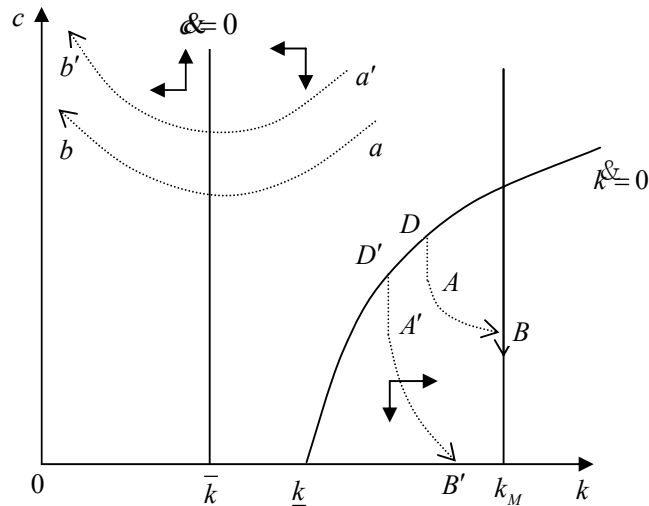
There exist three subcases of (k_0, c_0) ; (i) $k_0 \leq \bar{k}$, (ii) $\bar{k} < k_0 < \underline{k}$ and $\mathcal{R} \leq 0$, and (iii) $\underline{k} < k_0 < k_M$ and $\mathcal{R} \geq 0$.⁶⁾ For subcases (i) and (ii), as seen from the paths $a'b'$ and ab in figure 4, TVC (11) or TVC (12) is violated as

⁴⁾ From (8), we obtain $\lambda(t) = \lambda(0) e^{\int_0^t (\rho - r(\tau)) d\tau}$. By substituting this into TVC(9), we then observe $\lim_{t \rightarrow \infty} a e^{-\int_0^t r(\tau) d\tau} = 0$. The only case to check is $b \rightarrow \infty$ as $k \rightarrow 0$. However then, we obtain $\lim_{t \rightarrow \infty} a e^{-\int_0^t r(\tau) d\tau} = \lim_{t \rightarrow \infty} k e^{-\int_0^t r(\tau) d\tau} - \lim_{t \rightarrow \infty} b e^{-\int_0^t r(\tau) d\tau} = -\lim_{t \rightarrow \infty} b e^{-\int_0^t r(\tau) d\tau} = 0$. Hence in general $\lim_{t \rightarrow \infty} b e^{-\int_0^t r(\tau) d\tau} = 0$.

⁵⁾ NPG is equivalent to $\mathcal{B} b - r < 0$ as $t \rightarrow \infty$. From (2) and (3) $\mathcal{B} b - r < 0$ if and only if $-ex/b < 0$ if and only if $i + c - f(k)/b < 0$ as $t \rightarrow \infty$. Recalling $i \rightarrow 0$, $k \rightarrow 0$ and $c \rightarrow \infty$ along $a'b'$ as $t \rightarrow \infty$, $i + c - f(k)/b > 0$ for sufficiently large t , implying NPG is violated.

⁶⁾ We owe this classification to anonymous referee.

Figure 4 Case 2



discussed above.

For this case, just as Case 1 with $k_0 \leq \underline{k}$, we observe there exists no optimal path. For (iii) since $k_0 > \underline{k}$ holds if starting point are as D and D' , then the solution paths ABk_M and $A'B'$ are indicated by the phase diagram. However by sticking to points D and D' which are respectively above points A and A' forever, the larger amounts of value function is realized from these suboptimal paths. This implies that such paths as ABk_M and $A'B'$ are not optimal.

In short for this case, there exists no optimal path.

3. INTEREST RATE INCREASES WITH DEBT/CAPITAL RATIO

Next we consider the case where the borrowing interest rate r increases as the debt/capital ratio increases. Except the function $r = r(b/k)$, the model is the same as section 2. That is the function $r = r(b/k)$ is expressed as

$$r = r(b/k) \text{ with } r'(b/k) > 0 \text{ and } r''(b/k) < 0.$$

where b is the per capita bond (debt) issued by the home country and k is the amount of the home per capita capital. This assumption is made by e.g., Chatterjee, Sakoulis, and Turnovsky (2003) and Chatterjee and Trunovsky (2005). The law of motion of capital is (1), i.e.,

$$\dot{k} = i, \quad (1)$$

As before we obtain

$$\dot{a} = f(k) - c - r \cdot (k - a). \quad (4)$$

the law of motion of per capita net wealth a .

Here we assume again that the consumer takes the interest rate r as given, although it is dependent on debt/capital ratio. Then again the above maximization problem is solved by forming the current value Hamiltonian (5) and obtaining the first order conditions (6), (7)', (8) and TVC (the transversality condition) (9). Here (7)' is expressed as

$$f'(k) = r(b/k), \quad (7)'$$

From (6), (7)' and (8), we obtain again (10). From (7)' with $r = r(b/k) = r((k - a)/k)$, we obtain

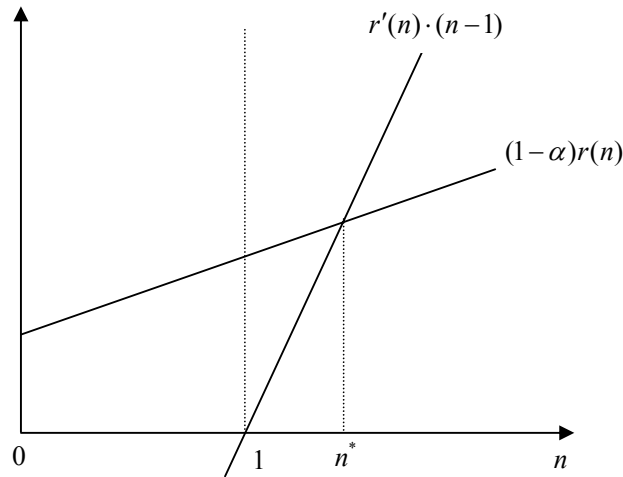
$$da/dk = (-f'' \cdot k + r' \cdot a/k) / r'. \quad (13)$$

To obtain the definite results, we specify r and $f(k)$ to be

$$r = r(0) + e^{\beta n} - 1, \quad (14)$$

where $\beta > 0$, $n = b/k$ and $r(0) = r_0$ is the international interest rate with

Figure 5



no debt.⁷⁾ Here $0 < r(0) < 1$ is assumed. $f(k)$ is of Cobb-Douglas so that

$$f(k) = k^\alpha, \quad 0 < \alpha < 1,$$

where α is the capital share. With these specification we obtain from (13)

$$da/dk > 0 \text{ if and only if } (1-\alpha)r + r' \cdot (1-n) > 0 \text{ if and only if } n < n^*,$$

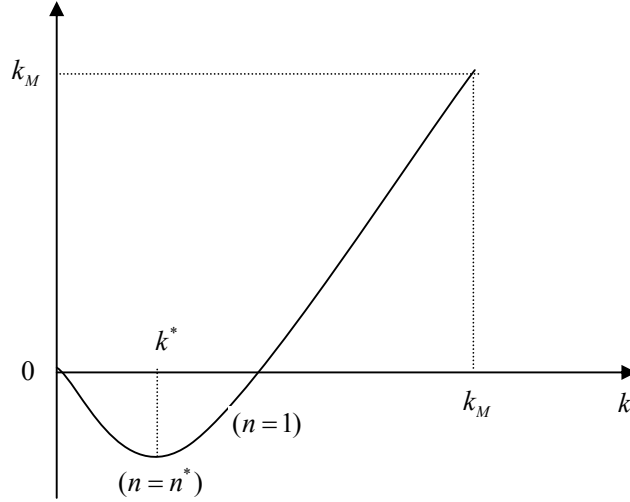
where $n = n^*$ is the value of n such that

$$(1-\alpha)r(n) + r'(n)(1-n) = 0. \tag{15}$$

In fact we observe from figure 5 that the slope of $r'(n) \cdot (n-1)$ curve is $r' + (n-1)r'' = r' + (n-1)\beta r' = (1 + \beta(n-1))r'$ and that of $(1-\alpha)r(n)$ is $(1-\alpha)r'(n)$, and hence the former's slope is higher than the latter's slope if $n > 1$ as drawn in figure 5, implying two curves intersect once at $n = n^* > 1$.

⁷⁾ We follow this specification to Chatterjee and Trunovsky (2005).

Figure 6



Now recalling $f'(k) = r(n)$, (7)' to imply $dn/dk < 0$, we obtain figure 6 where $k = k^*$ is defined to be the value of k such that $f'(k) = r(n^*)$ hold. As seen from figure 6,

$$da/dk > 0 \text{ if and only if } k > k^* \text{ if and only if } n < n^*.$$

Here we note that at $n \leq 1$, $a \geq 0$, and for $n > 1$, $a < 0$ follow. The maximum value of k , k_M is defined implicitly by $f'(k) = r(0) = r_0$, and hence at $k = k_M$, $a = k_M$ must hold. Furthermore by calculation, it is shown that $a \rightarrow 0$ as $k \rightarrow 0$.⁸⁾

Next we examine the function,

⁸⁾ From (7), (14) and $f(k) = k^\alpha$, we obtain $\alpha k^{\alpha-1} = r_0 + e^{\beta n} - 1$, where $n = b/k = 1 - a/k$.

This is rewritten as $a = k(1 - \frac{1}{\beta} \log(\alpha k^{\alpha-1} + 1 - r_0))$. Hence it suffices to show

$$\lim_{k \rightarrow 0} \frac{\log(\alpha k^{\alpha-1} + 1 - r_0)}{1/k} = \lim_{k \rightarrow 0} \frac{-k^2 \cdot \alpha(\alpha-1)k^{\alpha-2}}{\alpha k^{\alpha-1} + 1 - r_0} = \lim_{k \rightarrow 0} \frac{\alpha(1-\alpha)k^\alpha}{\alpha k^{\alpha-1} + 1 - r_0} = 0 \text{ in order to derive}$$

$a \rightarrow 0$ as $k \rightarrow 0$.

$$g(k) = r(n(k)) \cdot (k - a(k)), \quad (16)$$

the amount of the debt payment. By rewriting $g(k)$ as

$$g(k) = r(1 - a/k) \cdot (k - a).$$

We obtain $g'(k) = r' \cdot a \cdot b/k^2 + r - a'(r + r' \cdot b/k)$, where a' is obtained from (7)', (13) and $f(k) = k^\alpha$ as $a' = ((1 - \alpha)r/r') + a/k$.

Then $g'(k)$ is rewritten as

$$g'(k) = r \cdot (n\alpha - (1 - \alpha)r/r'). \quad (17)$$

Then $\text{sgn } g'(k) = \text{sgn}(n\alpha r' - (1 - \alpha)r)$.

Since the slope of $n\alpha r'$ curve is $\alpha(1 + n\beta)r'$ from $r'' = \beta r'$ and the slope of $(1 - \alpha)r$ is $(1 - \alpha)r'$, the former is always higher if $\alpha \geq 1/2$. If $\alpha < 1/2$, then from $\text{sgn } g'(k) = \text{sgn}(n\alpha r' - (1 - \alpha)r) - \text{sgn}(\alpha(1 + n\beta)r' - (1 - \alpha)r) = \text{sgn}(\alpha(1 + n\beta) - (1 - \alpha))$, we observe that although the slope of the latter is higher than the former for smaller n , there exists $n = \bar{n} = (1 - 2\alpha)/\alpha\beta$ such that $\alpha(1 + n\beta) > 1 - \alpha$ if and only if $n > \bar{n}$. (figure 8 corresponds to the case of $\alpha < 1/2$). Hence irrespective of $\alpha > 1/2$, there exists $n = \hat{n}$ such that $n\alpha r' = (1 - \alpha)r$ holds and

삭제됨: ,

$$n > \hat{n} \text{ if and only if } n\alpha r' = (1 - \alpha)r \text{ if and only if } g' > 0,$$

where \hat{k} is defined implicitly by $f'(k) = r(\hat{n})$. Then we obtain $g(k)$ curve as shown in figure 7. Here at $k = k_M$, $g(k) = 0$ follow from $b = 0$ at $k = k_M$. Furthermore recalling $a \rightarrow 0$ as $k \rightarrow 0$, we observe $g(k) \rightarrow 0$ as $k \rightarrow 0$.⁹⁾ Next by calculation we observe

⁹⁾ From (7), (14) and $f(k) = k^\alpha$, we obtain $0 \leq g(k) = r(k - a) = f'(k)(k - a) \leq \alpha f(k)$. Since $f(k) \rightarrow 0$ as $k \rightarrow 0$, we observe $g(k) \rightarrow 0$ as $k \rightarrow 0$.

$f(k) > g(k)$ if and only if $n < 1/\alpha$ if and only if $k > \underline{k}$,

Figure 7

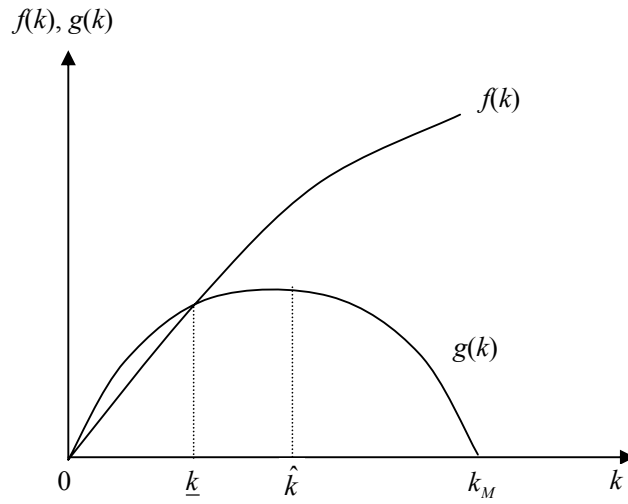
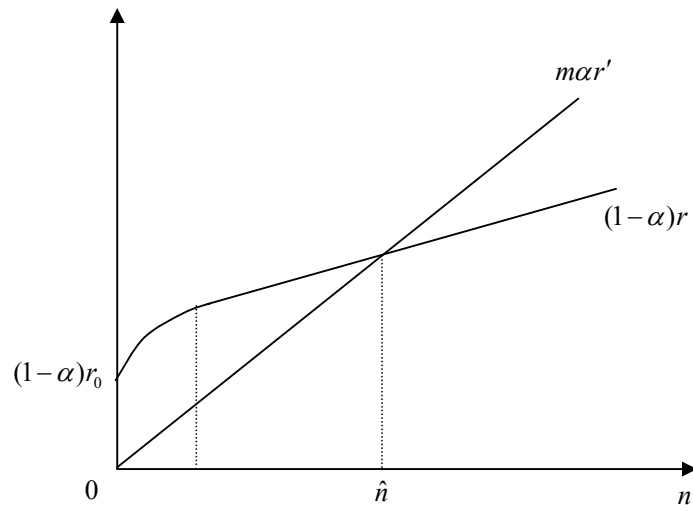


Figure 8



where \underline{k} is defined implicitly by $f'(\underline{k}) = r(1/\alpha)$.¹⁰⁾ This is shown in figure 7 (here we note that although $\underline{k} < \hat{k}$ is assumed in figure 7, it needs not so always. In fact the rest of the arguments goes through irrespective of $\underline{k} \geq \hat{k}$.)

Now let $k = \bar{k}$ be the value of k such that $f'(k) = \rho$ and hence $r = \rho$ holds.

Now we consider the possibility of the existence of the optimal path (k, c) , employing the phase diagram technique. First we consider the case where such an optimal path exists under an appropriate choice of (k, c) . This is equal to the case where the $\dot{c} = 0$ curve (the vertical line $k = \bar{k}$ in $k - c$ plane) and the $\dot{k} = 0$ curve (the upward sloping curve starting from $(\bar{k}, 0)$ in $k - c$ plane) intersect and the point of intersection $E(\bar{k}, \bar{c})$ is the stationary point of (k, c) where the optimal path (k, c) converges as $t \rightarrow \infty$. This case is Case 1 $\underline{k} < \bar{k}$ (i.e., $\rho - r_0 + 1 < e^{\beta/\alpha}$) as shown in figures 9.1 through 9.3. Case 1 is further subdivided into Case 1-1 ($k^* < \underline{k}$), Case 1-2 ($\underline{k} < k^* < \bar{k}$) and Case 1-3 ($\bar{k} < k^*$) corresponding to figure 9.1, 9.2 and 9.3 respectively. Here recalling $da/dk > 0$ if and only if $0 < k < k^* \Leftrightarrow \dot{c} > 0$ if and only if $k > k^*$, we observe that $\dot{k} > 0$ if and only if $\dot{c} > 0$ and that $\dot{k} < 0$ if and only if $\dot{c} > 0$ for $k < k^*$.

This implies that the horizontal arrowed line showing the direction of \dot{k} is reversed on the both sides of $k = k^*$ vertical line for all three cases as shown in figures 9.1 through 9.3. Now we can draw the solution paths of (k, c) based on the phase diagram for each case. For all three cases, the optimal path converging to $E(\bar{k}, \bar{c})$ is a saddle point path.

For Case 1-1 (figure 9.1) the optimal path of (k, c) starting from $k_0 \geq \underline{k}$ converges to E , but the solution path starting from (1) $k_0 \leq \underline{k}$, and (2) $k_0 \geq \underline{k}$ and $k < 0$ (if and only if $\dot{c} < 0$) is the one through which the home country goes bankruptcy eventually. In Case 1-1 where $k^* < \underline{k}$ (i.e., $(\alpha - \beta)e^{\beta/\alpha} > \alpha(1 - r_0)$

¹⁰⁾ By definition we observe $g(k) = r \cdot (k - a) = f'(k)(k - a) = f'(k) \cdot (k/\beta) \log(\alpha k^{\alpha-1} + 1 - r_0) = (\alpha/\beta)f(k) \cdot \log(\alpha k^{\alpha-1} + 1 - r_0)$ (see footnote 7) for its derivation). Hence $f > g$ if and only if $1 > (\alpha/\beta) \log(\alpha k^{\alpha-1} + 1 - r_0)$ if and only if $1 > \alpha n$ (from $\alpha k^{\alpha-1} + 1 - r_0 = f'(k) + 1 - r_0 = r + 1 - r_0 = e^{\beta n}$).

Figure 9.1 Case 1-1

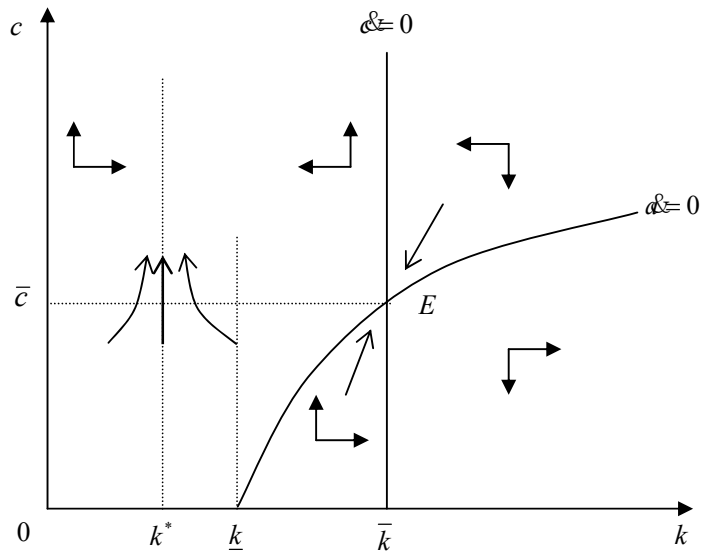


Figure 9.2 Case 1-2

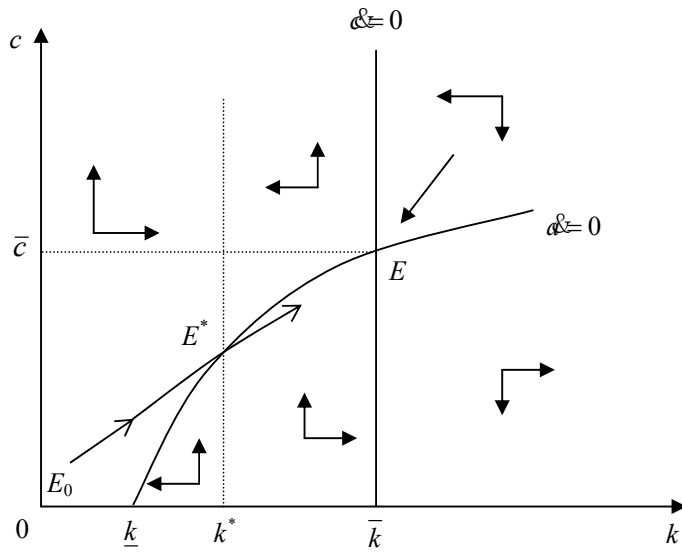
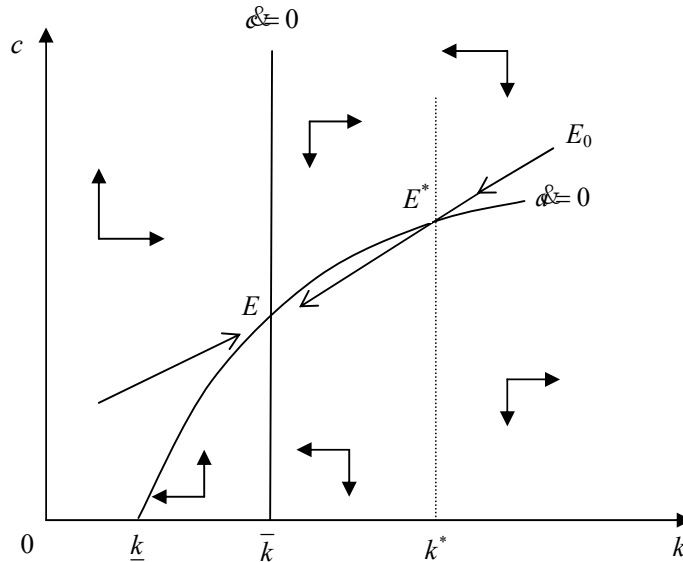


Figure 9.3 Case 1-3



implying $\alpha > \beta$) as seen from figure 9.1, if initial per capita capital is less than or equal to \underline{k} , then the solution path of (k, c) does not converge to the stationary state (\bar{k}, \bar{c}) but to (k^*, ∞) , and the home country goes into bankruptcy because she accumulates foreign debt. That is, along this latter path of (k, c) , NPG (No-Ponzi-Game) condition is violated again as seen by path $a'b'$ in figure 3 (Case 1 of section 2).

In short, the characteristics of the solution paths of (k, c) when the optimal stable saddle point path exists (i.e., Case 1) are, for Case 1-1, the possibility of the accumulation of foreign debt and the resulting bankruptcy when the initial k_0 is low.

For Cases 1-2 and 1-3, the economy can reach the stationary state E starting from any initial k_0 . For these cases the optimal path must pass through $E^*(k^*, c^*)$. Here k^* is defined implicitly by $f'(k^*) = r(n^*)$, and n^* by (15). Further c^* is defined by (4) with $\alpha\dot{c} = f(k^*) - c^* - r(n^*) \cdot (k^* - a^*) = 0$, where a^* is defined by $n^* = (k^* - a^*)/k^*$.

Here Case 1-1, $k^* < \underline{k}$ is equal to $(\alpha - \beta)e^{\beta/\alpha} > \alpha(1 - r_0)$ while in Cases

1-2 and 1-3 $\bar{k} < k^*$ is equal to $(\alpha - \beta)e^{\beta/\alpha} > \alpha(1 - r_0)$.¹¹⁾

Furthermore in Case 1-2 (resp. Case 1-3) $k^* < \bar{k}$ (resp. $k^* > \bar{k}$) is equal to $a'(k) > 0$ (resp. $a'(k) < 0$) at $k = \bar{k}$ (i.e., $\rho = r$), which is equal to $(1 - \alpha)r + r'(1 - n) > 0$ (resp. < 0) at $k = \bar{k}$ (i.e., $\rho = r$) or $(1 - \alpha)\rho/\beta > (\text{resp. } <)(\bar{n} - 1)e^{\beta\bar{n}}$ where $n = \bar{n}$ is the value of n at $k = \bar{k}$, i.e., $\bar{n} = (1/\beta)\log(\rho + 1 - r_0)$ from (14) with $\rho = r$.

Recalling here that Case 1, $\underline{k} < \bar{k}$ is equal to $\rho - r_0 + 1 < e^{\beta/\alpha}$, we note that the stable saddle point path of (k, c) is possible only if the discount rate ρ is low, initial world interest rate (interest rate without foreign borrowing) r_0 is high or β/α (=increasing rate of borrowing interest rate/capital-share) is high.

Next we consider Case 2. $\bar{k} \leq \underline{k}$ (i.e., $\rho - r_0 + 1 \geq e^{\beta/\alpha}$), the nonexistence of the stationary state. This case is further subdivided into Case 2-1. $k^* \leq \underline{k} \leq \bar{k}$, Case 2-2 $k \leq k^* \leq \bar{k}$ and Case 2-3. $\underline{k} \leq \bar{k} \leq k^*$, as shown respectively in figures 10.1, 10.2 and 10.3.

Here first we note that on the left (resp. right) side of the vertical line of k^* , $\dot{k} > 0$ (resp. < 0) if and only if $\alpha < 0$, as the Case 1, $\underline{k} < \bar{k}$. Then we can obtain phase diagram for all three cases.

First we consider Case 2-1 (figure 10.1).

Just as in Case 1-1, if the path starts from point a , a' or a'' , then c goes to infinity as $k \rightarrow k^*$ and $t \rightarrow \infty$. This implies that the home country goes bankruptcy, violating NPG just as Case 1-1. The path starting from point b arrives at B . However then this path is dominated by the suboptimal path be .

Here recalling that once $k = k_M$ is reached and the home country owes no debt, i.e., $b = 0$, then she receives the fixed world interest rate $r(0)$ when she turns into creditor, i.e., $b < 0$. Then we obtain as a first order condition

¹¹⁾ As seen from figure 6, $\underline{k} < k^*$ if and only if at $k = \underline{k}$ (i.e., $n = 1/\alpha$) $a'(k) < 0$ if and only if $(1 - \alpha)r + r'(1 - n) < 0$ at $n = 1/\alpha$ if and only if $\alpha r - r' < 0$ at $n = 1/\alpha$ if and only if $\alpha(r_0 - 1 + e^{\beta/\alpha}) - \beta e^{\beta/\alpha} < 0$ if and only if $(\alpha - \beta)e^{\beta/\alpha} < \alpha(1 - r_0)$.

$$f'(k) = r(0) = r_0, \quad (18)$$

if $b < 0$. Then (18) implies $k = k_M$ for $b < 0$ and hence $\dot{k} = i = 0$. This further leads to

$$\dot{c} = -\dot{b} = f(k_M) - c + r(0) \cdot (-b),$$

from the flow budget constraint (4) where $-b > 0$ is the asset held by the home consumer. Here we next consider the path starting from b' in figure 10.1. As discussed the above, once it reaches d on the vertical line after passing k_M , it moves down along dk_M toward $(k_M, 0)$, and reaches there. Here we note that such paths as starting from b and b' are dominated by the suboptimal path be and $b'e'$. That is, by choosing the suboptimal path to be such that starting from b and once reaches e stays there thereafter, the value of the value function

$W = \int_0^\infty u(c)e^{-\rho t} dt$ derived from this suboptimal path is clearly higher than the one obtained from the path beB . By the same reason, the path $b'e' dk_M$ is dominated by the path $b'e'$.

Then we observe that for such paths as starting points b and b' (i.e., $k_0 > \underline{k}$), solution paths exist, but these are not optimal path, and hence are never chosen, and that only such paths starting points as a , a' , and a'' are chosen, but these latter paths violate NPG (and hence TVC). In short there exists no optimal path for Case 2-1.

Next we consider Case 2-2 (figure 10.2).

This case is similar to Case 2-1. Such paths as beB and $b'e' dk_M$ are dominated by respectively the suboptimal paths be and $b'e'$. Hence only paths starting from a , a' , and a'' should be considered. If one path converges to points on the horizontal axis in a finite time $t = T < +\infty$, implying $c = 0$ at $t = T < +\infty$, then TVC, $H = 0$ at $t = T$ (11) is violated in view of (5) and (6) as discussed before. If another path converges to $(k^*, 0)$ as $t \rightarrow \infty$, then the first order condition (10) is violated as $t \rightarrow \infty$.

Figure 10.1 Case 2-1

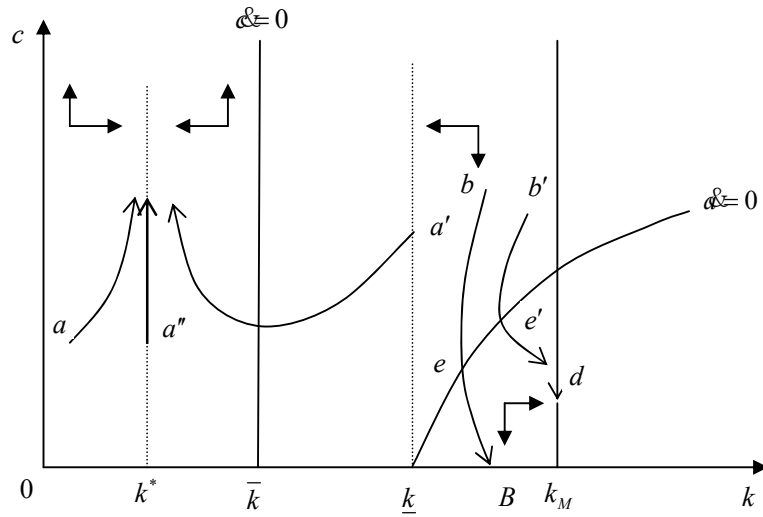


Figure 10.2 Case 2-2

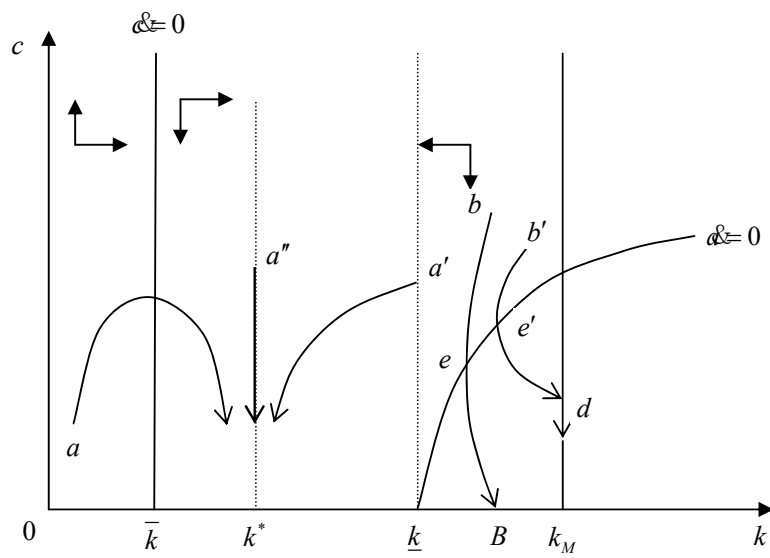
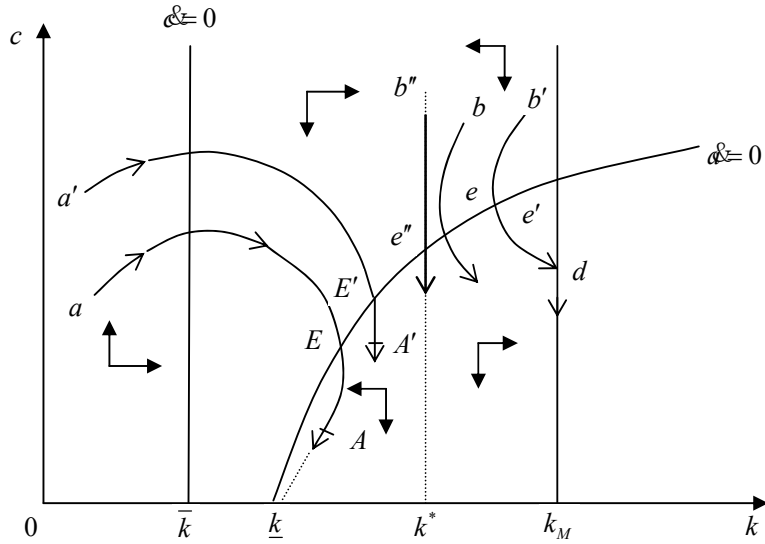


Figure 10.3 Case 2-3



In short there exists no optimal path for Case 2-2.

Lastly we discuss Case 2-3 (figure 10.3).

As discussed in Case 2-1 and 2-2, such paths as starting from b and b' are not optimal. Furthermore also such paths as starting from points a and a' and converging to some points on the horizontal axis are not optimal since these are dominated respectively by the suboptimal paths aE and $a'E'$ as discussed above.

In short there exists no optimal path for Case 2-3. Then we can obtain that there exists no optimal path for Case 2, i.e., $\bar{k} \leq \underline{k}(\rho - r_0 + 1 \geq e^{\beta/\alpha})$.

There exists several common characteristics of the results of section 2 ($r(b)$) and section 3 ($r(b/k)$). As seen from Case 1 of both sections, the stationary state E exists if positive consumption $c = \bar{c} = f(\bar{k}) - g(\bar{k})$ is realized at E where $k = \bar{k}$ is defined by $\rho = f'(k)$. Sufficient conditions for this to occur are (1) time discount rate ρ is not so large, (2) the foreign debt $b = \bar{b}$ at E is not so large where \bar{b} is defined by $f'(\bar{k}) = r(\bar{b})$ or

$f'(\bar{k}) = r(\bar{b}/\bar{k})$, which is possible if the rate of increase in r is not so high – i.e., the home country's risk premium does not increase so rapidly and (3) the level of productivity is not so low.¹²⁾

In other words if some of these conditions fail to be met, then the non-existence of the stationary state, and hence of the optimal path may occur. Especially if we recall that in some developing countries the time discount rates are high reflecting low level of consumption, risk premium is rather high due to accumulated foreign debt and level of labor productivity is low as a result of low rates of economic growth or of poor access to advanced foreign technology, then the plausibility of such a non-existence seems not negligible. Furthermore, even if the stationary state exists (i.e., Case 1 of section 2 and section 3), if the initial level of per capita capital k_0 is less than or equal to \underline{k} , then the economy cannot reach the stationary state for $r(b)$ (i.e., section 2) and for $r(b/k)$ (i.e., section 3) with $k^* < \underline{k}$. Again if we recall the capital scarcity of the developing countries, this possibility cannot be denied easily.

4. CONCLUDING REMARKS

The most significant difference between section 2 ($r(b)$) and section 3 ($r(b/k)$) lies in the likelihood of the occurrence of the optimal path toward the stationary state. In the former if the initial capital stock k_0 is less than or equal to \underline{k} , then the economy never reach the optimal path. However in the latter even if k_0 is less than \underline{k} , if k^* is larger than \underline{k} where k^* is the value of k at which the directions of k and a are reversed, then the economy can reach the optimal path as shown in Case 1-2 and 1-3 in section 3.

This further implies that the lending policy based on section 3 $r(b/k)$ seems

¹²⁾ Let $f(k, \theta)$ and $f'(k, \theta)$ where $\theta > 0$ being the technological parameter indicating the increase in the level of labor productivity. Then clearly increase in θ increases both \bar{k} and \bar{c} from $\rho = f'(k, \theta)$ and $\bar{c} = f(\bar{k}, \theta) - g(\bar{k})$.

better than section 2 $r(b)$ in the sense that the risk of bankruptcy is reduced – i.e., as far as the equilibrium point E exists (that is, $\underline{k} < \bar{k}$), even if bankruptcy occurs for the latter (i.e., $k_0 \leq \underline{k}$ for section 2), it never occurs for the former (section 2) if $\underline{k} < k^*$ holds. To our best knowledge this result is new, but we admit our model to be very simple – no government expenditure, no public investment, fixed time discount rate, and no investment adjustment costs etc., which are left for our next exercise. Also one natural way for generalization would be to extend into two-country model.

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